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TITLE GYROSCOPIC ANALOG FOR MAGNETOHYDRODYNAMICS

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GYROSCOPIC ANALOG FOR MAGNETOHYDRODYNAMICS

by

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ABSTRACT

The gross features of plasma equilibrium and dynamics in the ideal magnetohydrodynamics (MHD) model can be understood in terms of a dynamical system which closely resembles the equations for a deformable gyroscope.

INTRODUCTION

In the ideal magnetohydrodynamics (MHD) model, electrically neutral plasma convects like an adiabatic fluid that carries an embedded magnetic field. During convection, induced electrical currents flow instantaneously to oppose change of magnetic flux through each comoving surface. The resultant magnetic stresses alter the convective motion of the plasma by opposing bending of magnetic field lines.

We shall seek motions in three-dimensional MHD for which the velocity varies linearly in space. For such flows, time dependence factorizes out from all of the fluid variables in the Lagrange representation. The dynamical system which then results from Hamilton's principle closely resembles the equations for a deformable gyroscope.

Reduction to gyroscopic motion of fluid flow with linear velocity profiles was noted already in 1879 by Greenhill for circulation of a fluid of constant density within an ellipsoidal cavity. Before Greenhill, fluid flows with linear profiles had also been studied by Dirichlet, Dedekind, and Riemann, in connection with ellipsoidal figures of fluid equilibrium. The history and development of the latter topic is given with complete references by Chandrasekhar (1969). Rotating ellipsoidal fluid solutions are also treated in the classical texts on fluid mechanics by Basset (1888) and Lamb (1932).

More recently Parker (1957) has studied the expansion of a magnetic gas cloud which undergoes homogeneous dilation with linear velocity profiles, but which does not rotate or circulate. Likewise, Dyson (1968) has studied isothermal expansion and circulation of an ideal fluid whose velocity profile is linear and whose density profile is of Gaussian shape. Before Dyson, compressible fluid flows with linear profiles had also been noted by Ovsjannikov (1956). Subsequently Anisimov and Lysikov (1970) have found special solutions to Dyson's equations, that involve elliptic integrals for $\gamma = 5/3$ ideal gas.

In the next section we explain how time dependence factorizes out for MHD fluid flows with linear profiles, in the Lagrange representation. We then derive the equations of motion from Hamilton's principle, and analyze the resultant dynamical system for the time dependence of the flow.

The results provide an analogy between circulation of a magnetic fluid and angular momentum of a gyroscope. In this analogy, magnetic stresses produce elastic-like forces within the fluid which tend to restore both the circulatory motion and expansion of the fluid. In fact the equations for MHD with linear velocity profiles separate into two gyroscope equations which in general are coupled to each other by both magnetic stresses and deformations of shape. In the planar case with fixed elliptical boundary the equations reduce to the equation for a simple pendulum.

2. THE MHD EQUATIONS

In the Lagrange representation the particle paths are fundamental objects, and partial derivatives of the particle paths are basic dependent variables. The paths of fluid particles through fixed Eulerian space are given by vector functions $\underline{x}(t,\underline{x}^0)$ with initial conditions $\underline{x}(0,\underline{x}^0) = \underline{x}^0$, the Lagrange coordinate. The partial derivatives of the particle paths $\underline{x}(t,\underline{x}^0)$ produce the kinematical variables, velocity $\underline{v}$ and displacement gradients $\underline{F}_{ij}$ with components

$$v_i(t,\underline{x}^0) = \left.\frac{\partial x_i}{\partial t}\right|_{\underline{x}^0} = \dot{x}_i , \qquad F_{ij}(t,\underline{x}^0) = \left.\frac{\partial x_i}{\partial x^0_j}\right|_t \tag{1}$$

where subscripts t, $\underline{x}^0$ label the variables held constant in the partial derivatives.

In the Lagrange representation with Cartesian coordinates the equation of motion for ideal MHD is

$$\rho \frac{\partial^2 x_i}{\partial t^2} = -\frac{\partial}{\partial x_i}\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi} B^0_k \frac{\partial B_i}{\partial x^0_k} \tag{2}$$

where ρ, p are fluid density and pressure; $B_i(t,\underline{x}^0)$ without a superscript is magnetic field along a particle trajectory, and $B^0_i = B_i(0,\underline{x}^0)$ is its initial distribution. The equation of motion (2) follows from Hamilton's principle

$$\delta \int dt\, d^3x_0\, \rho^0 \left[\frac{1}{2}\dot{x}^2 - \varepsilon(\rho, s) - \frac{B^2}{8\pi\rho}\right] \tag{3}$$

for variations of the particle paths δx_k that vanish on the boundaries of the Lagrange domain of integration. The added notation in Hamilton's principle (3) defines $\rho^o = \rho(o,\underline{x}^o)$ as the initial density distribution, and $e(\rho,s)$ as the specific internal energy of the fluid which is a function of density, ρ, and specific entropy, s.

The variations of particle paths must be performed subject to the constraints of the following subsidiary conditions for MHD

$$\rho \det F = \rho^o = \rho(o,\underline{x}^o) \tag{4}$$

$$B^i = F_{ij} B^o_j / \det F \tag{5}$$

$$S(\rho, p) = S^o = S(\rho^o, p^o) \tag{6}$$

$$E(\rho, S) = E(\rho, S^o) \tag{7}$$

These subsidiary conditions impose respectively conservation of mass, Faraday's Law of magnetic induction, and the equations of state for adiabatic convection with the prescribed specific internal energy. In Faraday's Law and in the motion equation one uses Ampere's Law, curl $\underline{B} = 4\pi \underline{J}/c$, and Ohm's Law for the case of infinite electrical conductivity, $\underline{E} + \underline{v} \times \underline{B}/c = 0$, in order to eliminate current density, $\underline{J}$, and electric field, $\underline{E}$, in favor of magnetic field, $\underline{B}$, and particle velocity, $\underline{v}$.

Faraday's Law implies preservation of the divergence equation, div $\underline{B} = 0$, which thus may be regarded as an initial condition.

3. FACTORIZATION ANSATZ

By inspection of the subsidiary conditions for MHD one notes that time dependence factorizes in all of the variables, provided the displacement gradient is a function of time only,

$$F_{ij} = F_{ij}(t) \tag{8}$$

Once factorized, ideal MHD motion reduces to a dynamical system for the nine components of $F_{ij}(t)$. Hamilton's principle then acquires the matrix form

$$0 = \delta \int dt \left[\frac{1}{2} \mathrm{Tr}\,(\dot{F} I^{o} \dot{F}^{T}) - E(\det F) - \frac{\mathrm{Tr}(F S^{o} F^{T})}{\det F} \right] \tag{9}$$

with variations $\delta F_{ij}(t)$ and constants I^{o}, Π^{o}, S^{o} defined by integrals over the initial distributions of matter and magnetic fields,

$$\begin{aligned}
I^{o}_{kl} &= \int d^3x^{o} \rho^{o} x^{o}_{k} x^{o}_{l}, && \text{(initial moment of inertia)} \\
\Pi^{o} &= \int d^3x^{o}\, p^{o} && \text{(initial, integrated pressure)} \\
S^{o}_{kl} &= \int d^3x^{o}\, \frac{B^{o}_{k} B^{o}_{l}}{8\pi} && \text{(initial integrated magnetic stress)}
\end{aligned} \tag{10}$$

Variation of the action with respect to generalized coordinates $F_{ij}(t)$ produces the following motion equation when one chooses the polytropic adiabat $p\rho^{-\gamma} = p_{o}\rho_{o}^{-\gamma}$,

$$\ddot{F} I^{o} F^{T} = \mathbb{1} \left[\left(\frac{\Pi^{o}}{\det F} \right)^{\gamma - 1} + \frac{\mathrm{Tr}(F S^{o} F^{T})}{\det F} \right] - \frac{F S^{o} F^{T}}{\det F} \tag{11}$$

In the case that magnetic stress tensor S^{o}_{kl} is absent and the initial moment of inertia is unity, $I^{o}_{kl} = \delta_{kl}$, one recovers Dyson's equations for the spinning gas cloud.

4. MATHEMATICAL REMARKS

Before discussion of the motion of the fluid in detail, let us remark briefly on a mathematical aspect of the factorization Ansatz. The particle trajectories $\underline{x}(t,\underline{x}^0)$ arise from a smooth one-to-one mapping g_t that depends on time.

$$\underset{\sim}{g}_t : R^3 \times R \to R^3 \qquad \underline{x}(t,\underline{x}^0) = \underset{\sim}{g}_t(\underline{x}^0) \tag{12}$$

A smooth, one-to-one mapping whose inverse is also smooth is called a diffeormorphism, and such diffeomorphisms form a Lie group under functional composition.

The fluid velocity along each particle path is related to the curve of mappings g_t in the diffeomorphism group by

$$\dot{\underset{\sim}{g}}_t(\underline{x}^0) = \frac{d}{dt}\underset{\sim}{g}_t(\underline{x}^0) = \underset{\sim}{v}(t, \underset{\sim}{g}_t(\underline{x}^0)) \tag{13}$$

or equivalently

$$v(t,\underline{x}) = \dot{\underset{\sim}{g}}_t \circ \underset{\sim}{g}_t^{-1}(x) \tag{14}$$

Thus, the fluid velocity field along the particle trajectories is determined from the Lie algebra of vector fields associated with the diffeomorphism group.

Likewise, the displacement gradient is identified with $dg_t \cdot g_t^{-1}(\underline{x})$, the Jacobian of the map g_t. When this Jacobian is a function of only time, the particle paths become linear functions of the initial coordinates

$$x_i(t,\underline{x}^0) = F_{ij}(t)\, x_j^0 \tag{15}$$

where the time dependent matrix $F_{ij}(t)$ represents a general linear transformation of Cartesian coordinates, i.e., $F(t) \in GL(3,R)$. Thus, the diffeomorphism group for fluids specializes to the Lie group $GL(3,R)$ when the Jacobian of the evolution map depends only on time.

5. DYNAMICAL DESCRIPTION IN THREE DIMENSIONS

After the factorization Ansatz, the velocity along particle paths is given by

$$v_i(t, x) = [\dot{F} F^{-1}(t)]_{ij} x_j \tag{16}$$

or, in terms of Lagrange coordinates,

$$v_i(t, x^0) = \dot{F}_{ij}(t) x^0_j \quad . \tag{17}$$

The magnetic field also evolves by a linear transformation

$$B_i/\rho = F_{ij}(t) B^0_j/\rho^0 \tag{18}$$

and the density undergoes a time-dependent scaling

$$\rho = \rho^0/\det F(t) , \tag{19}$$

while the specific entropy is constant along each particle trajectory.

When this time dependence for $\rho, s, \underline{v}, \underline{B}$, is substituted into Hamilton's principle for MHD, there results the dynamical system (11) for the linear transformations $F_{ij}(t)$. Such transformations stretch the initial configuration of particles, and rotate the particle configuration relative to both Eulerian and Lagrange coordinate frames. Accordingly the displacement gradient $F_{ij}(t)$ may be decomposed into a matrix product

$$F = R_1 D R_2 \tag{20}$$

where R_1 and R_2 are orthogonal and D is diagonal. Each matrix depends upon time, and the decomposition $F = R_1 D R_2$ turns out to separate the motion into Eulerian rotations (R_1), dilations (D), and Lagrange rotations (R_2), the last of which represent circulatory motions of the fluid.

Upon substitution of the triple product $F(t) = R_1 D R_2$ into the dynamical system (11) one obtains the following separated equations

$$\dot{J} = 0$$

$$\dot{K} = \frac{2}{\det F}\left[S^{0}, F^{T}F \right] \tag{21}$$

$$\ddot{D}_i = -\frac{\partial}{\partial D_i} U(D)$$

where the skew-symmetric matrices J,K represent fluid angular momentum and circulation respectively

$$J = \int d^3x \, \rho \, (x_i v_j - x_j v_i) = F\dot{F}^{T} - \dot{F}F^{T} \quad (I = 1)$$

$$K = (v_{i,k} - v_{k,i}) F_{ki} F_{lj} = F^{T}\dot{F} - \dot{F}^{T}F \tag{22}$$

The bracket in the $\dot{K}$ equation is the matrix commutator, and the potential function $U(D)$ in the equation for the dilation matrix D is given by

$$U(D) = \frac{1}{4}\,\mathrm{tr}\,(\omega_1 L + \omega_2 N) + E(\det F) + \frac{\mathrm{tr}\, S^{*}D^{2}}{\det D} \tag{23}$$

with dynamical quantities ω_1, ω_2, L, N, S* defined by

$$\omega_1 = -\bar{R}_1 \dot{R}_1$$

$$\omega_2 = \dot{R}_2 R_2^{-1}$$

$$L = R_1^{-1} J R_1 = D^2 \omega_1 + \omega_1 D^2 - 2 D \omega_2 D \qquad (24)$$

$$N = R_2^{-1} K R_2 = D^2 \omega_2 + \omega_2 D^2 - 2 D \omega_1 D$$

$$S^* = R_2 S^o R_2^{-1}$$

The quantities ω_1, ω_2 are angular velocities of rotation and circulation respectively. The quantities L,N represent the angular momentum and circulation expressed in fixed, Eulerian coordinates. Finally $S^* = R_2 S^o R_2^{-1}$ is the magnetic stress tensor referred to the fixed Eulerian frame.

The equations of motion for J,K, and D first of all express conservation of fluid angular momentum, J. The circulation K is also conserved provided the magnetic stress tensor S^o_{ij} can be simultaneously diagonalized with the initial mass distribution I^o. However when the commutator $[S^o, F^T F]$ does not vanish, the circulation experiences a restoring torque due to magnetic stresses which are developed as the lines of magnetic field wind around themselves during fluid circulation. Finally the last equation for the dilation matrix D expresses the coupling between expansion of the fluid and its circulation and rotation. In the expansion potential U(D) the centrifugal, thermodynamical, and magnetic forces each are represented in conservative form, so energetic trade offs among them are clear.

6. COMPARISON WITH THE GYROSCOPE EQUATIONS

The reduced equations for J, K, and D express the fluid motion in the co-moving Lagrange frame. When transferred to the fixed Eulerian frame the resultant equations for $L = R_1^{-1} J P_1$ and $N = R_2^{-1} K P_2$ closely resemble the gyroscope equations expressed in body coordinates.

In body coordinates within a gyroscope that spins with angular velocity ω in a uniform gravitational field ($\underline{g} = (o,o,-g)$ in fixed coordinates) the equations of motion for angular momentum $M = I\omega$ take the matrix form

$$\begin{aligned} \dot{M} + [\omega, M] &= [g, C] \\ \dot{g} + [\omega, g] &= 0 \end{aligned} \qquad (25)$$

where C corresponds to the center-of-mass vector in the body, and the well-known correspondence, e.g., $C_{ij} = \epsilon_{ijk} C_k$ between $O(3)$ and R^3 has been used.

In order to compare with the gyroscope equations (25) one expresses the factorized equations for J,K in terms of their fixed-frame representatives L,N as

$$\begin{aligned} \dot{L} + [L, \omega_1] &= 0 \\ \dot{N} + [N, \omega_2] &= \frac{2}{\det D} [S^*, N^2] \\ \dot{S}^* + [S^*, \omega_2] &= 0 \end{aligned} \qquad (26)$$

Thus when $\omega_1 = 0 = \dot{D}$, for MHD fluid circulations with fixed shape and Eulerian orientation, the equation for fluid circulation in the fixed Eulerian frame N, has an analog with the gyroscope equation for angular momentum in the moving frame M, under the following identifications

$$
\begin{aligned}
M &\longleftrightarrow N \\
\omega &\longleftrightarrow -\omega_2 \\
g &\longleftrightarrow S^* \\
R &\longleftrightarrow 2D^2/\det D
\end{aligned}
\tag{27}
$$

The analog is not exact though, because S^*, D^2 are symmetric matrices while σ,P are skew-symmetric.

Thus, the equations for MHD motion with linear velocity profiles separate into two gyroscope equations which are coupled to each other by magnetic stresses and by deformations of shape. When the magnetic and material distributions can be simultaneously diagonalized, the angular motion becomes torque-free motion on $O(3) \times O(3)$, which can be further combined into geodesic motion on $O(4)$ by standard methods, see Holm (1981). In that case for motions with fixed shape the equations are completely integrable.

7. PLANAR FLOW: PENDULUM EXAMPLE

When constrained to rotate in a single plane the gyroscope reduces to a simple pendulum. Likewise the MHD circulation equation for N in fixed coordinates reduces to the equation for a simple pendulum in the case of planar MHD flow with a fixed elliptical boundary.

Consider planar circulation of a MHD fluid within a fixed ellipse whose principal axes (d_2, d_3) are aligned with the coordinate axes of the (x_2, x_3) plane. Because of the problem statement $\omega_1 = 0 = \dot{D}$, and the dynamical equation that remains is

$$\dot{N} + [N, \omega_2] = \frac{2}{\det D}[S^*, D^2] \qquad (28)$$

where the quantities N, ω_2, D, S* are given by

$$\begin{aligned} N &= D^2\omega_2 + \omega_2 D^2 = \dot{\varphi}(t)(d_2^2 + d_3^2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \omega_2 &= \dot{R}_2 R_2^{-1}(\varphi) = \dot{\varphi}(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ D &= \begin{pmatrix} d_2 & 0 \\ 0 & d_3 \end{pmatrix} \\ S^* &= R_2(\varphi+\alpha)\begin{pmatrix} s_2 & 0 \\ 0 & s_3 \end{pmatrix} R_2^{-1}(\varphi+\alpha) \end{aligned} \qquad (29)$$

with α the angle of rotation whereby S^0 is diagonalized,

$$R_2(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\tan\alpha = 2B_2^0 B_3^0 / [(B_2^0)^2 - (B_3^0)^2] \qquad (30)$$

Upon substitution of these definitions into the circulation equation, the pendulum equation emerges

$$\ddot{\varphi} = -A\sin(\varphi+\alpha) \qquad (31)$$

with natural frequency-squared A given by

$$A = \frac{d_2^2 - d_3^2}{d_2^2 + d_3^2} \cdot \frac{s_2 - s_3}{d_2 d_3} \qquad (32)$$

Thus, the gyroscopic analog for MHD provides an interpretation of planar circulation in terms of pendulum motion. The particle trajectories for fluid circulation are defined by

$$X_i = \left[\mathcal{I} R_2(\varphi(t))\right]_{ij} X_j^0 \qquad (33)$$

from which it follows that div $\underline{v}$ = div $\underline{\dot{x}}$ = 0. For this flow the density of the fluid is constant, and the magnetic field varies according to

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{d_2 d_3} \begin{pmatrix} d_2 & 0 \\ 0 & d_3 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} b_1^0 \\ b_2^0 \end{pmatrix} \qquad (34)$$

Thus the coordinates of the fluid particles and the magnetic field coordinates undergo pendulum motion within the ellipse.

At the boundary of the ellipse

$$\mathrm{Tr}\,(X^T \mathcal{I}^{-2} X) - 1 = 0 \qquad (35)$$

the normal components of both velocity and magnetic field vanish, provided the initial magnetic field b_i^0 is linearly related to Lagrange coordinates by

$$\begin{pmatrix} b_1^0 \\ b_2^0 \end{pmatrix} = b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \qquad (36)$$

where b is a constant of proportionality. Thus the condition of impermeability at the wall is satisfied for this solution, and both the velocity and magnetic field are divergenceless.

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